Dante and the 3-sphere

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We describe three different methods of visualizing the "closed universe" S^3, and point out language in Dante's *Divine Comedy* which suggests that he visualized his universe in the same ways, making his universe topologically S^3.

Physicists feel that physics is beautiful, but other people often greet this idea with incredulity, or at best thinly concealed tolerance. Indeed, it is very difficult to convey a sense of the beauty of physics to an audience which does not sense it already, while at the same time there is perhaps no deeper misunderstanding of science generally than the failure to see this quality. To illuminate what is beautiful in physics is not to offer light relief from the real subject matter, but to reveal something essential about the subject itself, something at its very heart.

The beauty of poetry, on the other hand, is said to be more generally accessible. Hence it may be worth pointing out an instance in which both poet and physicist grapple with the same issue and arrive at what is in essence the same vision. This example may give substance to the view that, at some deep level, poetry and physics are similar endeavors.

The example I have in mind is cosmology, as described by Dante in the *Divine Comedy*, and by Einstein in general relativity theory. It is not generally recognized that Dante's universe is explicitly non-Euclidean, perhaps because of our tendency to compartmentalize knowledge into subjects, and then to suppose that one subject has nothing to say about another. To state the thesis in a few words, Dante's universe is a three-dimensional sphere. His description of it is almost unbelievably apt and accurate, and his delight with this conceptual discovery is also rather movingly expressed. I will document this claim after a brief mathematical digression, which I believe even nonmathematicians will find painless. Einstein's preference for the spherical cosmology, on the other hand, is well known.[1]

I. INTRODUCTION

The belief that the earth must be round goes back at least to Aristotle, whose doctrine of "natural place" required a round earth at the center of the universe. This same model became central to Christian theology with the work of Thomas Aquinas, and it forms the cosmological framework for Dante's *Divine Comedy*.

The belief that the universe as a whole might be round (or more generally, curved) is a much more recent one. It seems to require mathematics of the 19th century (non-Euclidean geometry) even to formulate the notion.

It is therefore a considerable surprise to find, on closer reading, that Dante's cosmology is not as simple geometrically as it at first appears, but actually seems to be a so-called "closed" universe, the 3-sphere, a universe which also emerges as a cosmological solution of Einstein's equations in general relativity theory.

I came upon this suggestion about Dante and the 3-sphere in wondering how Dante would treat an evidently unsatisfactory feature of the Aristotelian cosmology when he, as narrator in the *Paradiso*, got to the "edge" or "top" of the universe. How would be describe the edge? It is the same problem every child has wondered about: unless the universe is infinite, it must (the argument goes) have an edge—but then what is beyond? Dante faces this very problem at the end of the *Divine Comedy* where he must describe the Empyrean not in terms of principles or abstractions, as the standard cosmology did, but as someone actually there.

The Empyrean is first seen and described in Canto 28 of the *Paradiso* in a passage I found, and still do find, surprising. The image is a 3-sphere—as good a description of one as I have ever seen anywhere. Dante thereby resolves the problem of the edge and at the same time completes his entire cosmological metaphor in a most astonishing and satisfactory way. I have since gathered that this passage is considered obscure by critics, but a relativist will at once see what is going on, and I will give the necessary mathematical background in detail in Sec. II.

This unexpected feature in a medieval cosmology makes an interesting addition to any discussion of "curved space," with obvious cross-disciplinary ramifications.

II. 3-SPHERE

The space we are going to describe is S^3, the 3-sphere, most easily represented as the sphere of some fixed radius R in four dimensions, that is, the set of (x,y,z,w) such that

$$x^2 + y^2 + z^2 + w^2 = R^2.$$

The non-Euclidean geometry of the closed Friedmann solution to Einstein's field equations is the one this sphere inherits from its embedding in Euclidean 4-space,[2] but we will actually be describing only its topology.

It is convenient to imagine S^3 as merely the next one in a sequence of familiar spheres of increasing dimension, denoted S^0, S^1, S^2, etc. These spheres S^n can be defined algebraically as the locus of points equidistant from the origin in $(n + 1)$-space.

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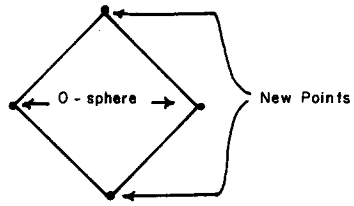
Fig. 1. Suspension of S^0 is S^1, a circle (topologically).



$$S^0\colon x^2 = R^2 \quad \text{(two points: } x = \pm R\text{)}$$

$$S^1\colon x^2 + y^2 = R^2 \quad \text{(circle)}$$

$$S^2\colon x^2 + y^2 + z^2 = R^2 \quad \text{(surface of ball)}$$

$$S^3\colon x^2 + y^2 + z^2 + w^2 = R^2 \quad \text{(3-sphere).}$$

Even if we cannot immediately imagine S^3, we can imagine its intersection with the hyperplane $w = w_o$. This is the set (x,y,z), such that

$$x^2 + y^2 + z^2 = R^2 - w_o^2,$$

which is a 2-sphere of radius $(R^2 - w_o^2)^{1/2}$ if $w_o^2 < R^2$, a single point $x = y = z = 0$ if $w_o^2 = R^2$, and empty otherwise. As w increases from $-R$ to R, then, we obtain the points of the 3-sphere as the points of a family of 2-spheres which grow in radius from zero to R and then shrink again to zero radius. (In a precisely analogous manner, the 2-sphere can be sliced up by planes into circles, and the circle can be sliced by lines into 0-spheres.[3])

A second method of visualizing the 3-sphere is, in a sense, inverse to the slicing method above. It is the "suspension" construction, and yields from each sphere the next *higher* in dimension.[4] The construction goes as follows: To suspend any space X, add two new points and form the "join" of these points to all points of X (i.e., connect every point of X to the two new points with lines or "strings"). The intuitive idea is to "hang" or "suspend" the space X like a hammock from the two new points. The resulting space is the *suspension of X*, denoted ΣX. As an example, we suspend the 0-sphere in Fig. 1. What we get is topologically a 1-sphere, or circle. It is not *geometrically* a circle, since it has straight segments, but it is *topologically* a circle. Similarly, the suspension of an n-sphere is an $(n + 1)$-sphere, i.e., $\Sigma S^n = S^{n+1}$. (See Fig. 2 for $\Sigma S^1 = S^2$.)

If we try to suspend the 2-sphere to get the 3-sphere, we must choose where to imagine the two new points. One new point can go inside the 2-sphere. The join of this new point to the 2-sphere gives a solid ball. The other new point must go outside to avoid a self-intersecting figure. In joining this new point to all points of the 2-sphere we run into a problem (see Fig. 3); not all the "strings" of the hammock which start at the new point and end at points close to each other on the 2-sphere can stay close together along their entire length. There must be a "part" somewhere, like the cowlick on a head of hair. The "part" is a flaw in the method of visualization and is not a property of the 3-sphere itself. It can be avoided only at the expense of some other flaw, like self-intersection. The next method circumvents this difficulty.

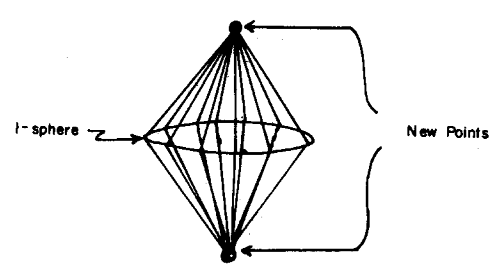
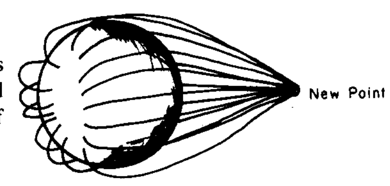


Fig. 2. Suspension of S^1 is S^2, a sphere (topologically).

Fig. 3. Suspension of S^2 is S^3, but this picture is flawed by the "part" on the left of S^2.



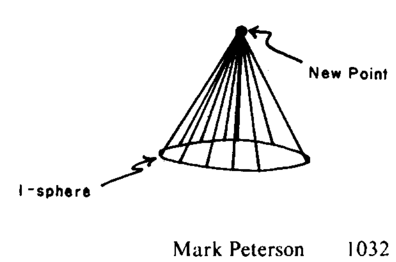
Our last method of creating higher-dimensional spheres from lower dimensional ones is by "gluing cones together." The cone on a space X, denoted CX, is the join of X with a new point. (The new point is the *apex* of the cone, and X is the *base*.) As an example, the cone on S^1 actually looks like a cone (see Fig. 4). As another example, the cone on S^2 is a solid ball if the apex is taken inside. Now imagine we glue CS^n to its mirror image by gluing corresponding points in the two copies of S^n (i.e., glue along the base). What we get is the suspension of S^n, i.e., S^{n+1}. In fact, forming two copies of CX and then gluing along X is just another way of describing the suspension construction (compare Fig. 2 and Fig. 5).

The two cones in this construction turn out to be the two hemispheres that make the sphere. The common base of the cones is the equator: it separates the two hemispheres, and is, of course, the sphere next lower in dimension.

This method gives us a very easy way to visualize S^3. It is composed of two solid balls, each being CS^2, which are the hemispheres, glued along their boundary, which is the equator S^2. Of course it is difficult to imagine it all glued up, but it is easy to imagine it glued over a representative piece of the boundary (see Fig. 6), and we can then just remember that each point in the boundary of one ball is really the same point as the corresponding point in the other ball, even if we choose to represent them as distinct points for ease of visualization. The resulting space is three dimensional, it has finite volume (namely twice the volume of one of the balls), yet it has no boundary; every point is interior. To check this, just note that an interior point of one of the balls is interior to the total space S^3; and also each point of the equator is interior to S^3; a neighborhood of a point in the equator intersects both hemispheres, but does not get outside S^3. To put it another way, an imaginary traveler leaving the right-hand CS^2 in Fig. 6 at the point P does not get outside the total space—rather, he enters the left-hand CS^2, just as a traveler on the earth leaves the northern hemisphere and enters the southern hemisphere when he crosses the equator. He would not even notice anything unusual about the point P—it is like any other interior point. Hence the traveler can never find an edge to his space—there isn't one.

The construction of S^3 by gluing cones together is like the construction of S^3 as a *manifold*, i.e., a space which is

Fig. 4. Cone on S^1.



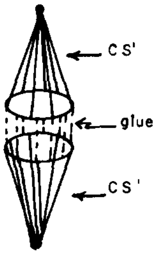


Fig. 5. Gluing together two copies of CS^1 along S^1 gives S^2.

locally like Euclidean space and can in fact be obtained by gluing together regions homeomorphic to Euclidean space ("coordinate patches") to make the whole. We could cover S^3 with two coordinate patches, each covering a hemisphere, and each being topologically a piece of ordinary three-dimensional space. (See Fig. 6.) The analogous construction of the 2-sphere in which the hemispheres are coordinatized by disks should help make this clear.

Our spatial universe might actually "be" S^3. What we are most sure of is that it is three dimensional and locally Euclidean. How the universe is connected globally, and whether it has finite or infinite volume, is not something a local measurement can ascertain, so far as we know. To explore this question we must obtain information from distant regions. In that respect Dante was more fortunate than we—he went to those distant regions and saw for himself!

III. DANTE'S UNIVERSE

In the *Paradiso* Dante describes his ascent sphere by sphere through the Aristotelian universe to the Primum Mobile. Beyond this is the Empyrean, the abode of God and the angels. The conventional picture of the Empyrean seems to have been rather vague, geometrically speaking. In diagrams of the universe, for example, it was represented by the border area, outside the Primum Mobile, often richly populated with angelic beings. Dante, however, endows the Empyrean with a detailed and precise geometric structure. This structure is described in Canto 28, as if seen from the Primum Mobile, as a bright Point representing God, surrounded by nine concentric spheres representing the various angelic orders. The details which follow leave the almost inescapable impression that he conceives of these nine angelic spheres as forming one hemisphere of the entire universe and the usual Aristotelian universe up to the Primum Mobile as the other hemisphere, while he is standing more or less on the equator between them, as if at the point P in Fig. 7 (see also Fig. 6). Taken all together, then, his universe is a 3-sphere.

In all, it is quite a stunning conception! In trying to make it plausible, I will point out how Dante makes verbal arguments which closely parallel the mathematics of Sec. II. He appears to be groping for a language to express an idea conceived intuitively and nonverbally, and in doing so actually gives in essence all three of our mathematical constructions.

Dante himself believed he was expressing something entirely new at this juncture. He asserts this by describing the difficulty of the notion as being like a knot that has grown tight, because no one has ever before tried to untie it.[5] There can be little doubt, however, that his new idea had no effect on cosmological thinking whatever—the 3-sphere in the *Paradiso* went unnoticed, or ununderstood. In recent times it has probably been dismissed by readers with less geometrical aptitude than Dante as mysticism.

To make the case, then, I first point out that Dante assumes from the outset that the nine angelic spheres and the nine heavenly spheres are analogous. In the notes to his translation, Ciardi makes this point by describing the angelic spheres as "a sort of counter-universe."[6] In fact, what interests Dante as narrator is a seeming *breakdown* in the analogy, about which Beatrice quickly reassures him.[7] The problem is that the various heavenly spheres revolve faster in proportion as they are bigger, while just the reverse is true of the angelic spheres: the innermost and smallest of these are revolving the fastest, and the outer ones are slower. Beatrice replies that if he will shift his attention away from the spheres' sizes to an intrinsic ranking they possess, he will see a marvelous consistency in the whole. The innermost angelic sphere turns faster than the other angelic spheres because it ranks higher, just as the Primum Mobile turns faster than the other heavenly spheres because it ranks higher. In other words, the spheres have a ranking, a "greatness," which does not necessarily correspond to their size (although for the first nine it does), but is rather indicated to the eye by their speed. This explanation strongly suggests our construction of the 3-sphere as sliced up into 2-spheres which first grow and then diminish in size, labeled by a fourth coordinate w, which simply increases. Indeed Dante has actually introduced such a fourth coordinate to label the spheres as they grow and diminish, namely their speed. In all our visualizations of the 3-sphere it was the second hemisphere, composed of the diminishing sequence of 2-spheres, which was hardest to fit into the model—Dante embeds the model in four dimensions, which does, as we know, solve the problem. His fourth dimension is speed of revolution. Of course he would never have said it that way, but it amounts to the same thing. The overall organization of the 2-spheres is that of a 3-sphere.

Dante's elation with this idea—a feeling we may readily share—has traditionally left readers somewhat puzzled. That is just another way of saying that if this passage is not taken as a description of the organization of 2-spheres into a 3-sphere, then it is hard to see what the point of it is.

Another stanza, worth pointing out, suggests the suspension construction, almost in a pun. It is translated by Dorothy Sayers as

"Observing wonder in my every feature
My Lady told me what I set below:
'From this Point hang the heavens and all Nature.' "[8]

Now it is not like a Euclidean space to "hang from a point," but it may be very natural for some other topological space. Indeed, topologists have invented this very notion to describe certain spaces via the suspension construction. In the above line, the Italian word is *depende,* which is etymologically almost identical with *suspension.* Considered as a suspension, Dante's universe is a remarkable extended metaphor: the earth, considered as S^2, is suspended between God and

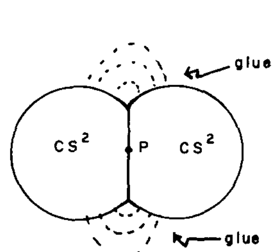


Fig. 6. Gluing together two copies of CS^2 along S^2 gives S^3.

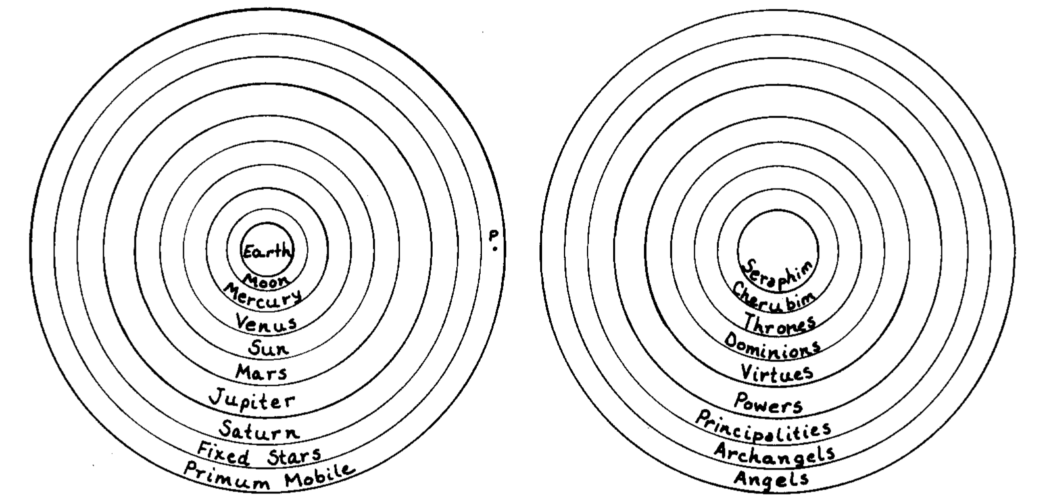


Fig. 7. The two semi-universes of Dante's universe.

Satan (at the center of the earth) to form the universe S^3.

Of course there is another, more abstract, meaning here. The heavens and all nature *depend upon* God for everything that they are. There is no doubt that this more abstract meaning is also intended. One may ask why Sayers, and Ciardi also for that matter, instead of using the word 'hang' do not simply translate *depende* as "depend." This might seem to the literal mind the most faithful rendering, and it would also call my own reading somewhat into question. In fact, I believe I know why Ciardi and Sayers are, so to speak, on my side. The concrete notion of "hanging down from" and the abstract notion of "depending upon" are closely fused in both Germanic and Romance languages. This fusion is less apparent in English than in any of the others—it takes a little classical training even to be aware of it, I think. In Italian, at least in the quoted passage, I believe the two meanings are present with about equal force; or it may be that, given the concreteness of the image being described, the concrete meaning even predominates. Thus the translation should be a concrete one—the metaphorical implications take care of themselves by associations built into the language. A very interesting counterexample is provided by Singleton, who translates the line as "On that point the heavens and all nature are dependent."[9] In justification of this more abstract rendering, Singleton cites a remarkable passage—Dante's own source for the line! It is apparently in Aristotle's *Metaphysics,* in a discussion of the "unmoved mover" as final cause and supreme good. Aristotle says "It is on such a principle, then, that the heavens and the natural world depend."[10] This is pure abstraction—there is nothing geometric about it. It is just what I find objectionable about the "top" of Aristotle's universe, and apparently just what Dante found inadequate. For Dante geometrizes this line, turning it into something concrete and visible; in short, he does here just what he is so famous for doing in so many other passages: fashions the perfect image. I believe Singleton's attention to the source has led him away from the poem, for it is not the similarity of these two lines, but their difference which is most interesting.[11] In making the line concrete Dante may well have been guided by etymology. This is why the line strongly suggests to me "suspension."

The manifold construction is suggested in the question Dante puts to Beatrice about the spheres' speeds:

"I must entreat thee further to explain
Why copy from its pattern goes awry . . ."[12]

Here "copy" and "pattern" refer, of course, to the angelic and heavenly spheres, respectively, and make it clear that one semi-universe is the example for the other. This is the essence of the manifold construction: to take two or more copies of something simple and put them together into a whole which is more complicated because of the way the copies are connected. As we have seen already, the question above is aimed at elucidating what the relation is between the two copies, i.e., the manifold structure.

When I first read Dante, I was impressed that he attempted to describe the stars as they would appear from the southern hemisphere (in Purgatorio). I am now enormously more impressed to find that he has done something like the same thing one dimension higher—described how a 3-sphere would appear from its equator. (This from a culture which, we occasionally are told, believed the earth was flat!) The overview of the 3-sphere is made completely explicit in the action of Cantos 27 and 28. In Canto 27 Dante looks down into the first semi-universe and sees the earth ("this little threshing floor") far below him. At the beginning of Canto 28 he *turns around* and *looks up* into the second semi-universe. This means the two hemispheres are positioned exactly as they should be. Consult Fig. 7 and imagine looking first one way, then the other, from the point P. Standing at the top of the Primum Mobile and looking first one way, then the other, is the way to see the entire universe in one sweep. The juxtaposition of these two actions just at this point certainly appears to be intentional and significant.

There are many ways that this cosmology "works" in the *Divine Comedy,* and it is one of the pleasures of reading the poem to discover them. I would not deprive the reader of these pleasures but will mention only one: the reflection symmetry of the 3-sphere in its equator. It is not just that God and Satan are represented as opposite poles, but even their neighborhoods, to use the mathematical word, are reflections of each other, the nine angelic circles finding a grotesque parody in the circles of the inferno. Considerations of symmetry must surely have played their part in Dante's creation of his image. As a matter of fact, the reappearance of S^3 in modern cosmology is largely via symmetry arguments too, although now it is a different sort of symmetry, homogeneity, and isotropy of the universe, which is taken as more compelling.

There are people who will never believe that physics is beautiful, and who perhaps still resent the way it demolished the medieval world view. They might be surprised to learn that modern physics can also illuminate the richness of the medieval imagination.

ACKNOWLEDGMENT

Explication of this idea has benefited from conversations with Ms. Marguerite Waller and with many students.

[1]See, for example, Einstein, *Relativity, The Special and the General Theory* (Crown, New York, 1931), pp. 128–134.

[2]Adler, Bazin, and Schiffer, *Introduction to General Relativity* (McGraw Hill, New York, 1975), p. 409.

[3]This method of slicing spheres will be familiar to many readers from Edwin A. Abbott's *Flatland* (Barnes & Noble, New York, 1963).

[4]The topological constructions we use can be found in, for example, John W. Keesee, *An Introduction to Algebraic Topology,* Brooks-Cole, Belmont, 1970), especially pp. 77 and 125.

[5]Dante, *Paradiso* (Harvard Univ., Cambridge, 1972), Canto 28, lines 58–60.

[6]John Ciardi in Dante, *The Paradiso* (Mentor, New York, 1970), notes to Canto 28, lines 21–36, p. 313.

[7]Dante in Ref. 5, lines 46–78.

[8]Dorothy Sayers and Barbara Reynolds in Dante, *The Divine Comedy 3, Paradise* (Penguin, Baltimore, 1962), p. 302.

[9]Dante, *The Divine Comedy,* translated, with a commentary by Charles S. Singleton, Bollinger Series LXXX (Princeton University, Princeton, N.J., 1975), *Paradiso 1,* p. 315.

[10]In Ref. 9, *Paradiso 2,* p. 450.

[11]I wish to thank an anonymous referee for raising this issue and suggesting that it ought to be discussed, and especially for drawing my attention to the Singleton line, which, it seems to me, epitomizes the whole matter.

[12]In Ref. 8, p. 302.